

**XXIII.** *The general Mathematical Laws which regulate and extend Proportion universally; or, a Method of comparing Magnitudes of any Kind together, in all the possible Degrees of Increase and Decrease.* By James Glenie, A. M. and Lieutenant in the Royal Regiment of Artillery.

Read March 6, 1777. **T**HE doctrine of proportion laid down by EUCLID, and the application of it given by him in his Elements, form the basis of almost all the geometrical reasoning made use of by mathematicians both ancient and modern. But the reasonings of geometers with regard to proportional magnitudes have seldom been carried beyond the triplicate *ratio*, which is the proportion that similar solids have to one another when referred to their homologous linear dimensions. This boundary, however, comprehends but a very limited portion of universal comparison, and almost vanishes into nothing when referred to that endless variety of relations, which must necessarily take place between geometrical magnitudes, in the infinite possible degrees of increase and decrease. The first of these takes in but  
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a very contracted field of geometrical comparison; whereas the last extends it indefinitely. Within the narrow compass of the first, the ancient geometers performed wonders, and their labours have been pushed still farther by the ingenuity and indefatigable industry of the moderns. But no author, that I have been able to meet with, gives the least hint or information with regard to any general method of expressing geometrically, when any two magnitudes of the same kind are given, what degree of augmentation or diminution any one of these magnitudes must undergo, in order to have to the other any multiply or sub-multiply *ratio* of these magnitudes in their given state; or any such *ratio* of them as is denoted by fractions or surds; or (to speak still more generally) a *ratio* which has, to the *ratio* of the first-mentioned of these magnitudes to the other, the *ratio* of any two magnitudes whatever of the same but of any kind. Neither have I been able to find that any author has shewn geometrically in a general way, when any number of *ratios* are to be compounded or decomposed with a given *ratio*, how much either of the magnitudes in the given *ratio* is to be augmented or diminished, in order to have to the other a *ratio*, which is equal to the given *ratio*, compounded or decomposed with the other *ratios*. To investigate all these geome-

trically, and to fix general laws in relation to them, is the object of this paper; which, as it treats of a subject as new as it is general, I flatter myself, will not prove unacceptable to this learned Society. It would be altogether superfluous for me to mention the great advantages that must necessarily accrue to mathematics in general, from an accurate investigation of this subject, since its influence extends more or less to every branch of abstract science, when any *data* can be ascertained for reasoning from. I shall, in a subsequent paper, take an opportunity of shewing how, from the theorems afterwards delivered in this, a method of reasoning with finite magnitudes, geometrically, may be derived, without any consideration of motion or velocity, applicable to every thing to which fluxions have been applied; and shall now proceed to the subject of this paper, after premising the two following definitions.

D E F I N I T I O N I.

**Magnitude** is that which admits of increase or decrease.

D E F I N I T I O N II.

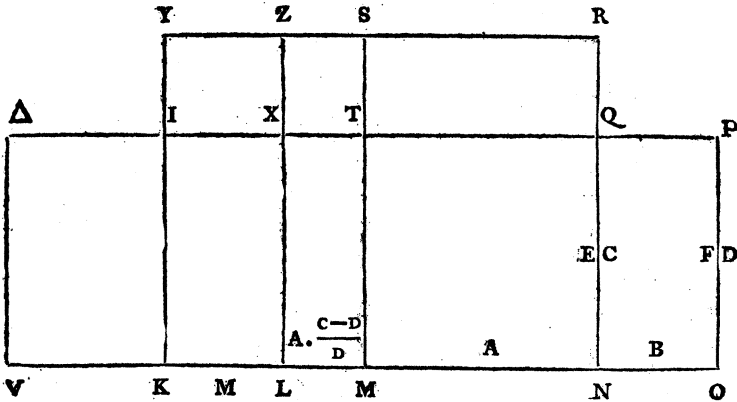
**Quantity** is the degree of Magnitude.

By magnitude, besides extension, I mean every thing which admits of more or less, or what can be increased

or

or diminished, such as *ratios*, velocities, powers, &c. As I shall frequently, for the sake of conciseness and conveniency, be obliged to make use of particular modes of expressing geometrical magnitudes, I here observe, once for all, that by such expressions as these  $A \cdot \frac{A}{B}$ ,  $A \cdot \frac{A-B}{B}$ ,  $A \cdot \frac{C-D}{D}$ ,  $A \cdot \frac{A-B}{B}$ , &c. I mean respectively a third proportional to B and A; a fourth proportional to B, A, and the difference of A and B; a fourth proportional to D, A, and the difference of c and D; a fourth proportional to B,  $A \cdot \frac{A-B}{B}$  and  $A-B$ , &c.

To proceed then in the order in which I first investigated these theorems; let A, B, C, D, E, F, G, H, &c. be any number of magnitudes of the same kind, taken two



and two from the first; and let MN, NO, NR, OP, respectively represent A, B, C, D. Let NR, OP, be drawn perpendicularly

pendicularly to vo, or otherwise if in the same angle; and let the rectangles or parallelograms MR, NP, be completed. Let LM be a fourth proportional to OP, MN and NR-OP; and let the rectangle or parallelogram LQ be completed.

Then (I 4. E. 6.) LT is equal to TR, and consequently LQ to MR. But (23. E. 6.) MR has to NP the *ratio* compounded of the *ratios* of MN to NO and NR to OP. Therefore (I. E. 6.) LN has to NO the *ratio* compounded of the *ratios* of MN to NO and NR to OP. But LN is equal to  $MN + MN \cdot \frac{NR-OP}{OP}$ , or  $A + A \cdot \frac{C-D}{D}$ , by construction. Whence it appears, that a magnitude of the same kind with A and B, which has to B the *ratio* compounded of the *ratios* of A to B and c to D, is expressed by  $A + A \cdot \frac{C-D}{D}$ .

In like manner let E, F, be represented by RN, OP, respectively, and let LK be a fourth proportional to OP, LN, and QR. Then (I 4. E. 6.) KX is equal to XR or TR and XS together. But since LN hath already been shewn to be equal to  $A + A \cdot \frac{C-D}{D}$ , LK is a fourth proportional to F, E-F, and  $A + A \cdot \frac{C-D}{D}$ ; that is equal to  $A \cdot \frac{E-F}{F} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F}$ . by construction. Wherefore KN being equal to LK+LN is equal to  $A + A \cdot \frac{C-D}{D} + A \cdot \frac{E-F}{F} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F}$ . And since XQ is equal to LR, KN has to NO a *ratio* compounded of

the *ratios* of LN to NO and NR to OP; that is, of the *ratios* A to B, c to D, and E to F. Therefore a magnitude of the same kind with A and B, which has to B the *ratio* compounded of these *ratios* is expressed by  $A + A \cdot \frac{C-D}{D} + A \cdot \frac{E-F}{F}$   
 $+ A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F}$ .

Again, if NR, OP, be supposed to represent G, H, respectively, and KV a fourth proportional to OP, KN, and QR; VQ will be equal to KR (I 4. E. 6.) and consequently VN will have to NO a *ratio* compounded of the *ratios* of KN to NO and NR to OP; that is, of the *ratios* A to B, C to D, E to F, G to H. But VK is by construction equal to

$$A \cdot \frac{G-H}{H} + A \cdot \frac{C-D}{D} \cdot \frac{G-H}{H} + A \cdot \frac{E-F}{F} \cdot \frac{G-H}{H} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}.$$

And this added to KN above found gives  $A + A \cdot \frac{C-D}{D}$   
 $+ A \cdot \frac{E-F}{F} + A \cdot \frac{G-H}{H} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} + A \cdot \frac{C-D}{D} \cdot \frac{G-H}{H} + A \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$   
 $+ A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ , for the magnitude of the same kind with A and B, which has to B the *ratio* compounded of the *ratios* A to B, C to D, E to F, G to H; whence the law of continuation is manifest.

The same conclusions may be derived from (E. 5.); so that no principle can be simpler or more geometrical than that here made use of.

Thus then these magnitudes will stand.

1.  $A + A \cdot \frac{C-D}{D}$ , when two *ratios* are compounded.

2.  $A + A \cdot \frac{C-D}{D} + A \cdot \frac{E-F}{F} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F}$ , when three are compounded.

3.  $A + A \cdot \frac{C-D}{D} + A \cdot \frac{E-F}{F} + A \cdot \frac{G-H}{H} + A \cdot \frac{C-D}{D} \cdot \frac{G-F}{F} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H} + A \cdot \frac{E-F}{F} \cdot \frac{G-H}{H} + A \cdot \frac{C-D}{D} \cdot \frac{E-F}{F} \cdot \frac{G-H}{H}$ , when four ratios are compounded, &c. &c.

By continuing this operation much farther, I found upon examination that the number of terms in which A is connected with the differences C-D, E-F, G-H, &c. taken one by one, two by two, three by three, &c. if  $p$  denote the number of ratios compounded, is expressed respectively by  $\frac{p-1}{1}$ ,  $\frac{p-1}{1} \cdot \frac{p-2}{2}$ ,  $\frac{p-1}{1} \cdot \frac{p-2}{2} \cdot \frac{p-3}{3}$ , &c. Thus if the ratio of A to B be supposed equal to the ratios of c to D, E to F, G to H, &c. respectively, these expressions will give the following ones.

$$1. A + \frac{2-1}{1} \cdot A \cdot \frac{A-B}{B}$$

$$2. A + \frac{3-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{3-1}{1} \cdot \frac{3-2}{2} \cdot A \cdot \frac{A-B}{B}^2$$

$$3. A + \frac{4-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{4-1}{1} \cdot \frac{4-2}{2} \cdot A \cdot \frac{A-B}{B}^2 + \frac{4-1}{1} \cdot \frac{4-2}{2} \cdot \frac{4-3}{3}$$

$A \cdot \frac{A-B}{B}^3$ ; for magnitudes of the same kind with A

and B, which have to B respectively the duplicate, triplicate, and quadruplicate ratio of A to B; where  $p$  is successively equal to 2, 3, and 4. And universally, by the same geometrical reasoning, it is found, that

$$A + \frac{p-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot A \cdot \frac{A-B}{B}^2 + \&c. A \cdot \frac{A-B}{B}^{p-1}$$

B such

B such a multiply ratio of A to B as is expressed by the number  $p$ .

In the reasoning above I fixed on B as the magnitude to which the rest were to be referred; but I might as well have fixed on A or any of the other magnitudes. Thus, for instance,  $B + \frac{p-1}{1} \cdot B \cdot \frac{B-A}{A} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot B \cdot \frac{B-A^2}{A} + \&c.$   $B \cdot \frac{B-A^{p-1}}{A}$  has to A such a multiply ratio of B to A as is expressed by the number  $p$ ; or A has to  $B + \frac{p-1}{1} \cdot B \cdot \frac{B-A}{A} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot B \cdot \frac{B-A^2}{A} + \&c.$   $B \cdot \frac{B-A^{p-1}}{A}$  the ratio of A to  $A + \frac{p-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot A \cdot \frac{A-B^2}{B} + \&c.$   $A \cdot \frac{A-B^{p-1}}{B}$  to B; that is, such a multiply ratio of A to B as is expressed by the number  $p$ . Each of these, indeed, I demonstrated separately from the same sort of geometrical reasoning; but for the sake of brevity I omit setting down these separate demonstrations, as they are both contained in general reasoning above, which furnishes likewise a great variety of other expressions, according as certain numbers of the ratios C to D, E to F, G to H, &c. are supposed to be respectively equal to, greater or less than, the ratio of A to B.

